

DISTRIBUTIONAL INTEGRAL TRANSFORMS

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Preface

There was a time when an undergraduate student, a postgraduate student or a research scholar of mathematics was expected to develop technique in solving problems that involved considerable computation; however, he was not expected to master theoretical subtleties such as real analysis or complex analysis, uniform convergence or uniform continuity. Now it is universally agreed that it is important for all students – whether future mathematicians, physicists, engineers or chemists – to grasp the basic theoretical nature of the subject. For, having done so, they will understand both the power and the limitation of the general theory and they will be better equipped to devise specific techniques to attack particular problem as they arise.

Schwartz's theory of distributions has two important effects in mathematical analysis. First of all, it provided a rigorous justification for a number of formal manipulations that had become quite common in the technical literature. The second and more important effect was that it opened up a new area of mathematical research, which in turn provided an impetus in the development of a number of mathematical disciplines, transformation theory, and functional analysis. However, the subject has remained pretty much in the realm of advanced mathematics, and only a few aspects of it have found their way into technical literature. To be sure, a certain type of distribution (in particular, the delta function and its derivatives) had been used in the physical and engineering sciences for quite some time before the advent of distribution theory.

This text has developed from my experience in teaching courses in distribution theory, integral transforms and functional analysis during the last one decade and is based on my research carried out on this subject during the same tenure,

and very specifically a large amount of contributions in which Dr. Deshna Loonker (who earned her Ph.D. in 2001 under my supervision) has collaborated with me, which resulted in many publications of research papers in national and international journals. Moreover, the present text is completely typed by Dr. Loonker and entire initial editorial work is done by her. In this text we have provided a comparatively elementary introduction to distribution theory and described the youngest generalization of the distributions, called Boehmians. There is also an introductory note on wavelet transforms and various methods employed to discuss wavelets. We have included three research papers to illustrate the applications of wavelet transforms to integrable and tempered Boehmians and investigated Plancherel theorem for wavelet transform. Integral transform methods have played an important role in the solution of initial and boundary value problems for partial differential equations.

In recent years an ever increasing number of textbooks have been devoted to the classical Fourier and Laplace transformations. The corresponding distributional transformations, to which our work is devoted, describes its potentialities and on the other hand we believe and hope that our work will help to popularize distributional transform analysis and raise an awareness among readers, who may be heterogeneous.

In writing this text I (PKB) have borrowed many notes from my classroom teaching and have been influenced by many sources. I have been benefited by religious contributions and collaborations of Dr. Deshna Loonker and, for various important discussions on this subject. This text also contains various lectures that I have delivered and papers that Dr. Loonker presented at many conferences and, out of the research projects that I have had worked upon during the past. We are thankful to various libraries from where we could collect many research materials, and to those who have given us opportunities to deliver lectures on this subject at various national and international conferences. Our thanks are due to the Faculty of the Department of Mathematics and Statistics of this University for their cooperation. We are thankful to Dr. Naseem Bhatia, Vice-Chancellor, J. N. V. University, Jodhpur (India), Prof. B.S. Paliwal, Dean, Faculty of Science and, the members of the

concerned committee, for granting a financial support in the form of UGC (India) Publication Grant under which this book is published. Authors are, indeed, thankful to Professor Lokenath Debnath, Department of Mathematics, University of Texas-Pan American, USA, for collaborating in joint research and to Professor S. L. Kalla, Department of Mathematics, Kuwait University, Kuwait, for having reviewed many of our research work. We extend thanks to Pawan Kumar, Scientific Publishers (India), Jodhpur, for painstaking efforts and personal interest in bringing out this text and to Mr. Rajesh Ojha, for expert typesetting of the complete original manuscript in the present form. We record thanks to Prof. Jyoti Das (Calcutta University, Kolkata, India) and Prof. Prem Chandra (Vikram University, Ujjain, India), the experts to review this Learned Research Work, for their comments and approval for its publication. Last, but not the least, I (PKB) recall my former students Dr. Dinesh Vyas, Dr. Yogendra Deora, Dr. Kuldeep Singh, Dr. Deepali Saxena (née Sinha), Dr. Abdul Malek (Yemen), Dr. Ahmad Galiz (Palestin), Dr. Abha Purohit, Dr. Fadal Mohsen (Yemen), Dr. Gamal Shenan (Palestine) who have worked with me during the past and earned their Ph.Ds and, S.K. Al-Omari (Jordan) who is working with me currently for the Ph.D. degree; all of them have excellent academic contributions.

One hundred and forty nine actually used references, in the traditional fashion, have been mentioned, which are arranged (numerically) in ascending order and authors in alphabetical order, with surname appearing first, followed by the initials, e.g. Hoskins, R.F., Nemzer, D., etceteras. References in the whole text, given as Pathak (107, 1998), Mallat (72, 1988), etceteras, be read as : name of the author of a paper or a book, which appears in the list of references as 107 or 72, is published in 1998 or 1988. The titles of books (monographs) and theses are typed *italics*. Sections and subsections have been clearly classified in each chapter, while equations numbered (2.61) or (3.32), will mean equation number 61 (or 32) of Chapter 2 (or 3).

Jodhpur (India)
21 July, 2005

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In collaboration with
Deshna Loonker

Abstract

The material of this text has been drawn from a number of different sources and is the outcome of some recent publications of the authors. The books by Zemanian, Vladimirov, Hoskins, Friedman, Pathak, Pandey, Chui and Walter are the standard references for the constructions of the present text. The theory of distributions and the generalized functions have been described carefully, while transform analysis is introduced to exhibit its adequate affinity with distribution theory. The prerequisites for this text and for those who wish to conduct research in this branch, are basic real and complex analysis, the functional Analysis and, the topology. Such as found in Rudin's *Principle of Mathematical Analysis*, Royden's *Real Analysis*, Vulikh's *A Brief Course in the Theory of Functions of a Real Variable*, *Complex Variables and Applications* by Churchill and Brown, Levinson and, Redheffer's *Complex Variables*, Copson's *Complex Variables* and Apostol's *Mathematical Analysis*, among very many other titles.

Generalized functions and distribution theory though synonym, yet differ in slightest meaning, and this text describes that difference. Studies in distribution theory began since the turn of the twentieth century and ever since it raised numerous curiosities among mathematicians and physicists. Integral transforms and the distribution theory, though two different strings, cultivated many potential contributions and concepts, together, for the harmonical development of mathematics and physics. The Fourier, the Mellin, the Stieltjes and the Mehler-Fock integral transforms have been studied over the years, and in the present text we have endeavoured to treasure such developments. The two approaches to the theory of distributions, the functional analysis approach and the sequential approach have been described through potential theories and authors' publications. The introduction of δ -function by Paul Dirac in

1927 in quantum mechanical studies, having certain properties, was pointed out by mathematicians that, from the purely mathematical point of view this definition of the δ -function is meaningless, of which Dirac was aware too. The entities that are to be analyzed are specified as functions on some class of testing functions (as mapping of the testing-function class into the space of real or complex numbers).

This text has four chapters and a possible extensive list of references. There will, probably, be many more references which are not included, which is because we might have not seen them. We may, in addition, mention that such kind of work on the distribution theory and transform analysis will first time occupy place in the library of our department, which otherwise is rich in topics on pure and applied mathematics. In a course to understand the philosophy of pure and applied mathematics and enrich the Schwartz theory of distribution, **Chapter 1** describes (in a possibly easy language) the concepts through illustrations and definitions. We have synthesized all terminologies regarding integral transforms and the methods employed to use them for generalized functions (the distribution). In one of the subsection of this chapter we have been able to show a remarkable difference between the terms, Distribution and Generalized functions. Tempered distributions and ultradistributions have been explained, while rarely used concept of distribution, the sequential approach, occupies a substantial space in this chapter. Theory of Boehmians, a younger generalization of the theory of distribution, is explained with all possible fundamentals, so as to reach to all common researchers (because we do not have any text until now to describe Bohmian and Bohmian space). This chapter concludes with an account of recent developments.

Association of Schwartz distribution theory and transform analysis is the theme of **Chapter 2**. Gegenbauer polynomial forms the Gegenbauer transform and properties of which are known. There are two sections of this chapter; in the first, we have proved and extended the Gegenbauer transform to a class of distribution (called it as the distributional Gegenbauer transform) defined on the weak topological spaces and then an operational formula is obtained, with a discussion on its

application to solve a differential equation involving generalized functions. Abelian theorems for the generalized and distributional generalized Mehler-Fock transformation are established as the initial and final value theorems. The investigations of this chapter have been published in the *Proceedings of the First National Conference of SSFA (ed. P. K. Banerji) in 2001*.

Chapter 3 attempts yet another matured discussion on sequential approach to tempered distributions, the ultradistribution and the distributions of slow growth for integral transforms. There are three distinct parts of this chapter, dealing with sequential approach for the tempered distribution for the Hilbert transform (*appeared in the Far East Journal of Mathematical Sciences in 2001*), ultradistribution of Mellin transform (*published in the Proceedings of the National Academy of Sciences, India, in Vol. 70, 2000*) and distributions of slow growth (*published in the Journal of the Indian Academy of Mathematics, Vol. 23, 2001*). Sequential approach is preferred over functional approach, because it has no rigidity of topological concepts. This part has a brief introduction to tempered distribution for Hilbert transform and two very small theorems explain our aim. The *second part* begins with a prologue on Mellin and Fourier transform relation, the space \mathbf{C}^* and its dual \mathbf{C}' with properties, and contains five exhaustive theorems, solved in simple mathematics and references properly quoted. The *third part* introduced the relation between Fourier and Mellin transforms and then appropriate testing functions spaces and distribution of slow growth are employed to prove conditional theorems for the stability of distribution of slow growth for the Mellin transform. Mapping properties of the test function of Mellin transform are also obtained.

Whereas the preceding three chapters surveyed, very extensively, the finer fundamentals of distributions (generalized functions) and various methods by which the integral transforms could be associated with the theory of distributions, the **Chapter 4** is an exception. Two sections of this chapter deals with two different concepts, accommodated in a possible manner and are addressed to very many mathematicians for further explorations. Mellin transform of tempered Boehmians

(published in *Bulletin of Applied Mathematics and Physics of the Politechnique Univ., Romania, Vol. 3, 2001*) is the theme of the *first section* of this chapter, introduction of which is partly borrowed from Chapter 1, while the central result (believed to be new in the literature according to the concerned referee of the Bulletin) is based on the concept : *The Mellin transform of tempered Boehmian is a Schwartz distribution. Second section* describes and surveys the wavelet transform (integral wavelet transform) which in the recent time has become a very common language among pure and applied mathematicians and provides a fascinating interface between physics and mathematics. This survey article was presented at the National Conference of the Jadavpur University, Kolkata, during March 2001. This section concludes with the inclusion of three useful research work by the authors on the applications of wavelet transforms.

Inspite of our desire to include, all authors of this branch, we have not been able to do so. This is not deliberately done, may be we have not employed their results in our work or our ignorance about their work. However, the typescript is checked properly, but in case of any typographical error authors are fully responsible.

Report on the Learned Research Project

*The Learned Research work, entitled DISTRIBUTIONAL INTEGRAL TRANSFORMS, submitted by Professor P. K. Banerji, Department of Mathematics, J. N. V. University, Jodhpur, is an exhaustive survey of the researches that led to the theories of **distributions, generalized functions and transform involving them**. The author has taken all possible measures to indicate the entire existing directions of investigations done till date, including even the latest of them, viz., **Boehminas**. His lucid language has made the write-up a reading of great pleasure.*

*The significant choice of delta function of Paul Dirac (1927) as the eye-opener for the theories to be developed indicates the author's enviable power of exposition. The vague idea of the delta function, used abandonedly by the physicists, was aptly rigorised to the mathematical entities, called **distributions**, simply as **mappings of a testing function class** (to be defined) into the space of real or complex numbers by Laurent Schwartz (1950). And the **generalized functions** were developed by G. Temple (1953) as "limits" of sequence of functions, where "limits" are to be defined properly with the assistance of a class of functions, called **test functions**. The author has nicely pictured the similarities and the dissimilarities that exist between distributions and generalized functions.*

*Various choices of the testing function space obviously lead to various types of distributions, of which **tempered distributions** and **ultradistributions** claim much significance. The reason is simple : they provide suitable platforms for the **generalizations of Fourier transforms, Stieltjes and Mellin transforms**.*

A very carefully arranged exposition of these topics has surely enriched this “Learned Research Work”

T. K. Boehme (1973) considered the linear space G of all continuous functions and a subspace S of G ; a map $*$: $G \times S \rightarrow G$ is defined satisfying some conditions. A sequence of elements of S is called a delta sequence if it satisfies some prescribed conditions. If Δ denotes the family of all delta sequence and ξ denotes the family of all sequences of elements from G , Boehme defined an equivalence relation in $\xi \times \Delta$. Each equivalence class thus defined is known as a **Boehmian**. An extremely useful subclass of the class of Boehmians is the class of **tempered Boehmians**. Their importance is exhibited by the fact that **every distribution is the Fourier transform of a tempered Boehmian**. The well-known link between Fourier transform and Mellin transform obviously leads to a similar result on Mellin transform.

Many mathematicians have devoted much of their valuable time to **transform analysis**. Accordingly a good number of transforms have been defined. The main hurdle lies in the corresponding **inversion-theorem** of a given function. As shown by the tempered Boehmians, the answers to such problems are likely to be found in the theory of distributions. That it is actually so has been beautifully depicted by the author of this Learned Research Work.

The representations of distributions as boundary value of analytic functions was studied by Luszczki, Z. and Zieleny, Z. (1961) and Tillman, H. G. (1961). These led to the Abelian Theorem for Distributional transforms, presented nicely in the Learned Research Work.

The author has also included : “Wavelet transform” in his Learned Research Work in order to overcome the limitations of Fourier series, viz., its nonapplicability to nonperiodic functions and its inability to depict local behaviour. D. Gabor (1946) considered **Windowed Fourier transform** by using Gaussian Distribution function as the **window function** introduced in the Fourier transform. Careful choice of the window function is expected to enable one to eradicate the limitations of Fourier series. It is to be investigated whether distributions may offer

their candidature for such window functions in order to achieve desired results.

A meticulous extensive study of the existing literature on the subject has produced such a neatly written Learned Research Work. It will be of enormous help to every researcher of the fields of distributions and transform analysis. They will have several directions of research exposed to them for further exploration. Surely, everybody working in these fields will love to possess a copy of this Learned Research Work.

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Report for Learned Research Work

*The book entitled **DISTRIBUTIONAL INTEGRAL TRANSFORMS** has been extensively examined by me and found it extremely useful to research workers and may even be treated as a reference book to Post Graduate students. All the **four chapters** are very clearly written containing reputed work of the authors and I admire the learned collection. A chapter on **Wavelet transforms** and **Boehmians** is totally a new concept and contains very elegant results.*

I, therefore, have no hesitation in recommending this work for publication under UGC programme. I have known Dr. Banerji's academic excellence for many years.

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