

**B.D. Tikkiwal**

# T-Classes of Linear Estimators and the Theory of Successive Sampling

*Edited by*

**G.C. Tikkiwal**

$$\hat{T}_7 = \sum_{r=1}^n \beta_{ro}^{s_t} X_r$$

$$\hat{T}_2 = \sum_{i \in s_t} x_i / \pi_i$$

$$\hat{T}_g = \sum_{i \in s_t} \beta_i^{s_t} x_i$$

$$\hat{X}_h = (1 - \phi_h) Z_h + \phi_h X_{hn''}$$

# T-Classes of Linear Estimators and the Theory of Successive Sampling

*This book is an edited version of the Lecture Notes on Advanced Theory of Sample Surveys of the author **Prof. B.D. Tikkiwal**. The lecture notes were out-growth of lecture material that evolved over many years of author's teaching and research work.*

**G.C. Tikkiwal**

Jai Narain Vyas University  
Jodhpur



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**Dr. G.C. Tikkiwal**

Professor of Statistics

Dean, Faculty of Science

and

Head, Department of Mathematics & Statistics

Jai Narain Vyas University,

Jodhpur - 342 001, India

E-mail: gctikkiwal@yahoo.com

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The responsibility for the facts stated, opinions expressed/and conclusions/ reaches is/ are entirely that of the editor and that the University accepts no responsibility for them.

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## Prof. B.D. Tikkiwal (1926-2004)



Prof. B.D. Tikkiwal was one of the pioneering statisticians and leading experts on the conceptual and theoretical frame work of survey sampling.

As founder and Head of the Department of Statistics at the University of Rajasthan, Professor Tikkiwal was the main force behind starting statistics courses at both the undergraduate and postgraduate levels in other universities of the state as well. Today, his students are employed in university and government positions throughout India and abroad.

Professor Tikkiwal was born on January 15, 1926, in Jaipur the capital city of Rajasthan State. He obtained his early education in Jaipur and his B.Sc. and M.Sc. degrees from Maharajas College, Jaipur. While obtaining his B.Sc. degree, he was awarded the Maharana Udaipur Gold Medal, and while obtaining his M.Sc., he was awarded the Chancellor's Gold Medal. He continued his studies at the Indian Agricultural Statistics Research Institute, New Delhi, 1948-51, where he was awarded the Randhawa Gold Medal based on his thesis *Theory of Successive Sampling*. In 1952-54, he studied Mathematical Statistics at Columbia University. In 1955, he earned his Ph.D. in Experimental Statistics from the University of North Carolina at Raleigh.

After working as Associate Statistician at the University of North Carolina, he joined Karnataka University, Dharwar, India in June 1956 and worked there for six years as Reader and Head. He subsequently moved to the University of Rajasthan, Jaipur, where he established the Department of Statistics, formerly the School of Sample Surveys, and worked there until his retirement in 1986. During his tenure, he held positions as Reader, Professor, and Senior Professor. He also held important administrative positions as founder and Head of the Department of Statistics (1962-74), Director of the School of Social Sciences (1967-68), Dean of University Cluster of Colleges, Jaipur (1975-76), Chairman, Examination Reforms Cell and acting Vice-Chancellor on two separate occasions.

Professor Tikkiwal earned national and international recognition for his contributions in sampling theory. He was President of the Statistics Section of the Indian Science Congress in 1977. He was Principal Speaker at the session on rotation and other resampling schemes at the International Statistical Institute's (ISI) Session in Manila, Philippines, in 1979 and a participant at the session on conceptual and theoretical framework for

survey sampling at the ISI's 43rd Session in Buenos Aires, Argentina, in 1981. In 1985 he participated in ISI's 45th Session in Netherlands. He went to the USSR (1969) under a Cultural Exchange Program for Senior Professors, and to Nigeria (1980-82) as Visiting Professor and Head of the Department of Statistics, University of Ilorin. He visited and lectured at numerous universities on many occasions throughout his career. He remained active even after his retirement in 1986 until the time of his death. In 1989, he was awarded an Emeritus Fellowship by the University Grants Commission, India, and in 1990 he established the Institute of Developmental Research and Statistics at Jaipur. He participated in International conferences at Warsaw (Poland) in 1992, at Washington, D.C. (U.S.A.) in 1998 and at Montreux (Switzerland) in 2000.

Professor Tikkiwal was a life member of the Indian Society of Agricultural Statistics, a member of the Institute of Mathematical Statistics, a member of the International Statistical Institute, and a Council Member of the International Association of Survey Statisticians. He was proud of his election to Sigma  $\chi$ , the American society devoted to promotion of research in science. He organized numerous conferences and courses for university and college teachers.

Apart from his main field of Sampling Theory, Professor Tikkiwal did extensive work by way of application of the theory to education and to some extent agriculture. He associated himself with many sample surveys in agriculture, education, and socio-economic fields. As a member of the Statistical Committee for the Indian Council of Agricultural Research at New Delhi (1964-67) and as a member of the working group on Small Area Statistics (1988-90) set up by the Central Statistical Organization of the Government of India, he gained first-hand knowledge of practical problems in data collection and their analysis. This experience helped him in providing a realistic basis to his work in theory, which, in turn, won him international recognition in this area of study. He also engaged in vital research methodology problems of providing reliable estimates of food production at District and Panchayat Samiti (small area) levels in India. This methodology problem was based on the SICURE model developed by him, which he presented at the International Conference on Small Area and Survey Designs in Warsaw, Poland, in 1992.

Prof. Tikkiwal credited his professional success to being influenced early on by Dr. G.S. Mahajani, then Vice-Chancellor of the present day University of Rajasthan, who was instrumental in leading him on to his career in statistics. He was also influenced by the guidance of Indian Statisticians such as P.V. Sukhatme, R.D. Narain and G.R. Seth as well as American and British Colleagues such as T.W. Anderson, D.G. Horvitz and D.R. Cox.

*(as contributed by Prof. T.W. Anderson, Stanford University in the International Statistical Institute Newsletter Vol. 30, No. 1 (88), 2006)*

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## Preface to First Write-up

The lecture notes,\* on advanced theory of sample surveys, are out-growth of lecture material that evolved over many years of my teaching and research in this area. It was in the year 1959-60 that I included for the first time in my lectures, on Advanced Sample Surveys course, for M.A./M.Sc. Statistics students of Karnataka University, the material on T-classes largely based on Horvitz and Thompson's paper (1952). Since this paper had material only on sampling without replacement, I redefined  $T_2$ -class estimator in the manner given in Chapter 3 and asked the students to work out the details similar to those given by Horvitz and Thompson. Dr. Prabhu Ajgaonkar, then a Ph.D. research scholar working under my supervision in sampling on Successive Occasions, was also attending these lectures alongiwth the students. He came out with some results which were so encouraging that I asked him to switch over to the area of T-classes. The early results obtained in this area were reported in the Annals of Mathematical Statistics (Prabhu Ajgaonkar and Tikkiwal, 1961) in an abstract form. Some of these results were concerning  $T_2$ -class.

Tikkiwal (1965) reviewed the earlier work, broadly in the area of T-classes of Horvitz and Thompson (1952), Sen (1953), Yates and Grundy (1953), Prabhu Ajgaonkar and Tikkiwal (1961), Rao-Hartley-Cocharan (1962). He further defined in this paper extended  $T_2$ -class for multi-stage designs and after developing its general theory showed how the estimators due to Hansen and Hurwitz (1943), Horvitz and Thompson (1952), Sukhatme and Narain (1952), Durbin (1953), Ecimovic (1956) lie in the extended  $T_2$ -class. The results given in this paper provide most of the material of Chapter 3.

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\* A copy of the original lecture notes is available with the Library of International Statistical Institute at 428 Prinses Beatrixlaan, P.O. Box 950, 2270 A Z Voorburg, The Netherlands

The material in Chapter 3 alongwith the related material on sampling with varying probabilities in standard text books like that of Sukhatme and Sukhatme (1970) and the material on sampling theory, such as unbiased ratio method of estimation and non-sampling errors, as a left over material from one semester basic course on sample surveys, can form a one good semester course on Advance Theory of Sample Surveys. This observation is based on the experience of running this course in the University of Rajasthan for last four years under the semester system, introduced from 1972 and for couple of years under annual system.

The material in part (a) of Chapter 1 together with the two appendices at the end is meant to indicate how the first few lectures on a basic course in sample surveys should be organised to provide a sound base to this course. The material in parts (b) and (c) of Chapter 1 on the partition of sample space and then on the definition of T-classes of linear estimators and the technique of unordering and combined unordering (Tikkiwal, 1972), has been included, in much detail only this year, for the first time in my lectures on a semester course on sample surveys being given at the University of Rajasthan for some years. By the technique of unordering and combined unordering one can get all the results obtained otherwise by Basu (1958) and Pathak (1964) using the concept of sufficiency in survey sampling. This technique provides additional results in the process.

The material in Chapters 2, 4, 5 and 6 on the remaining T-classes along with the material on the theory of successive sampling in Chapter 7 in meant for one specialised sample survey course in a semester scheme. It should follow the one semester advanced course referred to above. The material in Chapter 7 on the theory of successive sampling should be supplemented from standard text books such as those by Sukhatme and Sukhatme and by Cochran. These text books deal with the theory under certain assumptions for an infinite population, where as this chapter deals with the theory under less restrictive assumptions for a finite population, there by making it possible to obtain the main results in these text books as a special case of the results presented in the chapter.

Some typed copies of all the above material except that given in Chapter 6 have been available in the Department of Statistics, University of Rajasthan for the use of concerned students and teachers. This material is presented in the various chapters with suitable modifications. The material in Chapter 6 on  $T_5$ ,  $T_6$  and  $T_7$ -classes has been added to the course from last year only. This

material is based on the paper, by Bhargava and Tikkiwal (1973), which gives two new classes  $T_5$  and  $T_6$  based on the suitable partition of the sample space (Tikkiwal, 1972) used for developing the technique of unordering and combined unordering. The typed copies also contained the material on unified theory of sampling (Godambe, 1955) and on sufficiency in survey sampling. The latter material has been omitted in view of the addition of the material on unordering and combined unordering; and the former one has been abridged and improved upon in Chapter 6 in view of the fact that Godambe's general linear estimator is not general enough and is only a sub class of  $T_7$ -class.

Because of my other pre-occupations at the time of writing these lecture notes and because of other environmental limitations there must have remained many weak points including mistakes of various types, omission etc. I shall be grateful to the Readers for pointing them out, so that I can improve upon these notes.

In preparing the above material, I have been ably assisted by Dr. C.L. Agarwal and Dr. G.C. Tikkiwal, the then two research fellows under my supervision. Dr. G.C., in fact, delivered lectures to M.A./M.Sc. Statistics students on the material in Sections 1 to 7 of Chapter 3, while it was being written. This arrangement, in which some one directly under the author rather than the author himself delivers the lecture worked much better in gauging the difficulties in presenting such material to the students. The stencilling etc. was done mainly by Mr. P.L. Sharma. My thanks go to all the above persons and to others who might have helped me in the task knowingly or unknowingly.

University of Rajasthan,  
Jaipur, India  
November, 1977.

**B.D. Tikkiwal**

## Preface to Second Write-up

Since bringing out these lecture notes in 1977, it was gradually felt, that to add to clarity the notations used by the author (1972) in describing the partition of sample space need to be improved upon. This has been done in Section 1.1 of Chapter 1 and consequential changes are made in subsequent five chapters on T-classes.

Another major change in this write up is to take away the material in part (a) of Chapter 1 together with the two appendices at the end of the first write up. This material was meant to indicate how a basic course in sample surveys should be organised. This material has now been considerably enlarged and appears separately (Author, 1985).

Soon after the first write up of these lecture notes, the paper by Bhargava (1978) appeared on the applications of the technique of unordering and combined unordering originally given by the author (1972). Some of these are now included in Chapter 2. In Chapter 1, an improved estimator for SRSWR is given using the technique of combined unordering. However, the estimator of the variance of the estimator based on distinct units in case of this sampling scheme is not included in these lecture notes, as it needs improvement.

From the lecture notes on the basic course, we know that all the estimators, in use, for equal probability design, lie in  $T_1$ -class. From Chapter 3 of these lecture notes, we know that most of the estimators, in use, for varying probability designs for uni-stage and multi-stage sampling, lie in author's  $T_2$ -class. Thus, the study of these two classes is a must for any one interested in the theory of sample surveys. The study of these classes as a whole is further necessiated by noting the fact that their theory (Tikkiwal, 1985) generates estimators for different sample designs. For example, if we are using Midzuno scheme of sampling with or without replacement, we get estimators for this sampling scheme in  $T_1$ -class,  $T_2$ -class and Godambe's general class discussed in Section 7 of Chapter 6. Some of these estimators can be further improved upon by the author's technique of unordering and combined unordering given in Chapter 7.

The theoretical treatment to the theory of successive sampling in Chapter 7 remains more or less the same as in the first write up. However, for the first time we find in a text book (Sukhatmes and Asok, 1984) treatment to the study of a finite population on successive occasions. But still the theoretical treatment in these lecture notes is far more exhaustive and complete.

My thanks go to the University Grants Commission and the University of Rajasthan for making it possible to organise in October-November, 1985 the All India Advanced Institute in Sample Surveys and Inference for training teachers in the new approach to the teaching of courses in sample surveys developed during the course of my teaching these courses for about three decades at the University of Rajasthan and my teaching at the University of Ilorin, Nigeria for about two and a half years as Visiting Professor there. As part of the programme of this Institute, these lectures were revised and up-dated.

I also thank Shri V.N. Rai and Dr. Diwakar Khare for their help in getting ready the manuscript in typed form and then photo-copied or stenciled. Lastly, but not the least, I thank Shri P.L. Sharma for his pains-taking efforts in typing or stenciling the manuscript.

University of Rajasthan,  
Jaipur, India  
July, 1986

**B.D. Tikkiwal**